Phys202 - Dynamics - Spring 2007 HW3 Solutions

1 Problem 12.14

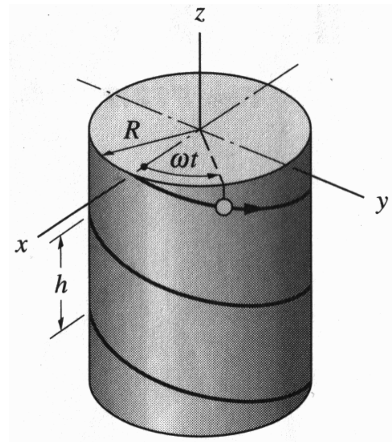

Figure 1: Helix for Problem 12.14.

When a particle moves along the helix shown, the components of its position vector are

$$x = R\cos\omega t \qquad y = R\sin\omega t \qquad z = -\frac{h}{2\pi}\omega t$$

where ω is constant. Show that the velocity and acceleration have constant magnitudes and compute their values if $R = 1.2\,\mathrm{m}$, $h = 0.75\,\mathrm{m}$, and $\omega = 4\pi\,\mathrm{s}^{-1}$.

Solution: In rectangular coordinates, take the derivatives to get the components of the velocity

$$v_x = \dot{x} = R\left(-\omega\right)\sin\omega t \qquad v_y = \dot{y} = R\omega\cos\omega t \qquad v_z = \dot{z} = \frac{-h\omega}{2\pi}$$

The magnitude is from the 3-D pythagorean theorem

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(R\omega\right)^2\left(\sin^2\omega t + \cos^2\omega t\right) + \left(\frac{h\omega}{2\pi}\right)^2} = \sqrt{\left(R\omega\right)^2 + \left(h\omega/\left(2\pi\right)\right)^2}$$

The components of the acceleration are

$$a_x = \dot{v}_x = -R\omega^2\cos\omega t \qquad a_y = \dot{v}_y = -R\omega^2\sin\omega t \qquad a_z = \dot{v}_z = 0$$

and the magnitude of the acceleration is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = R\omega^2$$

Notice that this is equalivalent to $a = v^2/R$ because $v = R\omega$.

Substituting the requested values,

$$v = \sqrt{\left(\left(1.2\,\mathrm{m}\right)\left(4\pi\,\mathrm{s}^{-1}\right)\right)^2 + \left(\frac{\left(0.75\,\mathrm{m}\right)\left(4\pi\,\mathrm{s}^{-1}\right)}{2\pi}\right)^2} = 15.2\,\mathrm{m/s} \qquad a = R\omega^2 = \left(1.2\,\mathrm{m}\right)\left(4\pi\,\mathrm{s}^{-1}\right)^2 = 189\,\mathrm{m/s}^2$$

2 Problem 12.19

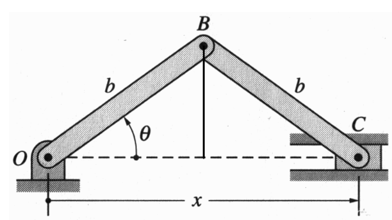

Figure 2: Diagram for Problem 12.19. The vertical line dropped from B splits x into two parts, and it's easy to see that $\frac{x}{2} = b\cos\theta$.

For the mechanism shown, determine (a) the velocity $\dot{x}$ of the slider C in terms of θ and $\dot{\theta}$; and (b) the acceleration $\ddot{x}$ of C in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$.

Solution: The coordinate x is twice the adjacent angle of a right triangle.

$$x = 2b\cos\theta$$

So, the velocity in this 1-D motion is the derivative, remembering the chain rule $\frac{df(\theta)}{dt} = \frac{df(\theta)}{d\theta}\frac{d\theta}{dt}$:

$$v = \dot{x} = 2b\left(-\sin\theta\right)\dot{\theta}$$

And, taking the derivative again:

$$a = \dot{v} = -2b\left(\sin\theta\,\ddot{\theta} + \dot{\theta}\cos\theta\dot{\theta}\right)$$

3 Problem 12.47

If the block is released from rest in the position shown, determine its velocity when it hits the floor.

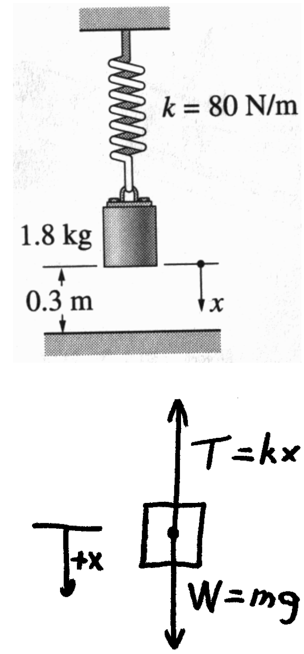

Figure 3: Mass on a spring for Problems 12.47 and 12.48. When the 1.8 kg block is in the position shown, the attached spring is undeformed.

Solution: This is a 1-D problem, so $a = \dot{v}$. Because the x coordinate is labeled positive in the downward direction, that is also the positive direction for velocity, acceleration, and force.

$$\sum F = W - T = ma = m\dot{v}$$

The tension in a spring is $T = kx$ and it is positive (i.e. in the direction drawn) when the spring gets longer. Filling these values into the equation, we see we have to use the technique for $a(x)$.

$$\begin{aligned}
mg - kx &= ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \\
(mg - kx)\,dx &= mv\,dv \\
\int (mg - kx)\,dx &= \int mv\,dv \\
mgx - \frac{1}{2}kx^2 &= \frac{1}{2}mv^2 + C_1 \\
E_0 &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 - mgx
\end{aligned}$$

This is the **conservation of energy** equation with $E_0 = -C_1$ being the total energy. By keeping the mass times acceleration instead of dividing it out, each term after integration is in the units of energy. (The gravitational potential energy is $mgh = -mgx$ because x was defined positive downward.) This method is a little more general than simple conservation of energy, though, because it can handle non-conservative forces.

Now that we have a relationship between v and x, we can use the initial conditions to determine the constant. Here, when $x = 0$, $v = 0$, which means $E_0 = 0$. Then we set $x = 0.3\,\text{m}$ and find v.

$$\begin{aligned}
\frac{1}{2}mv^2 &= mgx - \frac{1}{2}kx^2 = (1.8\,\text{kg})\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(0.3\,\text{m}) - \frac{1}{2}\left(80\,\frac{\text{N}}{\text{m}}\right)(0.3\,\text{m})^2 = 1.697\,\text{J} \\
v &= \sqrt{\frac{2\,(1.697\,\text{J})}{(1.8\,\text{kg})}} = 1.373\,\text{m/s}
\end{aligned}$$

4 Problem 12.48

In the system from Problem 12.47 above, the block is pulled down until it touches the floor and then released from rest. Calculate the maximum height (measured from the floor) reached by the block.

Solution: The diagram and force diagram are the same as above. Instead of re-integrating as above, since this is the exact same geometry, start with:

$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 - mgx$$

This time, $v = 0$ when $x = (0.3\,\text{m})$so

$$E_0 = \frac{1}{2}\left(80\,\frac{\text{N}}{\text{m}}\right)(0.3\,\text{m})^2 - (1.8\,\text{kg})\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(0.3\,\text{m}) = -1.697\,\text{J}$$

The maximum height occurs when $v = 0$, so

$$-1.697\,\text{J} = \left(40\,\frac{\text{N}}{\text{m}}\right)x^2 - (17.66\,\text{N})\,x$$

$$40x^2 - 17.66x + 1.697 = 0$$

$$x = \frac{17.66 \pm \sqrt{17.66^2 - 4\,(40)\,(1.697)}}{2\,(40)} = 0.221 \pm 0.079\,\text{m} = 0.142\,\text{m}$$

The distance from the floor is $\boxed{3 - x = 0.158\,\text{m}}$.

5 Problem 12.116

The free length of the spring that is attached to the 0.4 lb slider A is L_F. If the slider is released rom rest when $x = 8\,\text{in}$, calculate its initial acceleration. Also calculate $v(x)$.

Solution: The tension in the spring is $T = k\,(L - L_F)$ and the tension is positive (in the direction drawn) as L gets larger. Since the motion is 1-D, we can use the simple $a = \dot{v}$. The length of the spring is $L = \sqrt{x^2 + y^2}$, using y for the constant $3\,\text{in}$ vertical distance. Also, note that as x increases, the object moves to the left, which is opposite what we are used to. Newton's Second Law is then

$$\begin{aligned} F &= -T_x = ma \\ a &= -\frac{T}{m}\left(\frac{x}{L}\right) = -\frac{k}{m}\frac{(L - L_F)}{L}x \end{aligned}$$

The initial conditions given are $x_0 = 8\,\text{in} = 0.6667\,\text{ft}$, $k = 6\,\text{lb/in} = 72\,\text{lb/ft}$, $m = \frac{w}{g} = \frac{0.4\,\text{lb}}{32.2\,\text{ft/s}^2} = 0.0124\,\text{slug}$, $L_F = 0.5\,\text{in} = 0.04167\,\text{ft}$, and $L = \sqrt{x^2 + y^2} = \sqrt{(8\,\text{in})^2 + (3\,\text{in})^2} = 8.544\,\text{in} = 0.712\,\text{ft}$. Plugging these in, I get $\boxed{a = -3645\,\text{ft/s}^2}$. It is negative because x is decreasing.

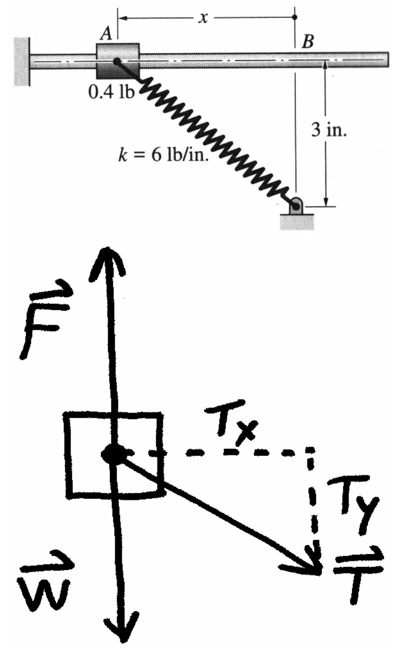

Figure 4: Diagram and Force Diagram for problem 12.116.

To solve for $v(x)$, we need to find the equation of motion, i.e. $a\,(x)$ and integrate.

$$\begin{aligned} ma = mv\frac{dv}{dx} &= \frac{-k\,(L - L_F)\,x}{L} = \frac{-kx\left(\sqrt{x^2 + y^2} - L_F\right)}{\sqrt{x^2 + y^2}} \\ mv\,dv &= \frac{-kx\left(\sqrt{x^2 + y^2} - L_F\right)}{\sqrt{x^2 + y^2}}\,dx \end{aligned}$$

Integrating this looks pretty hard, but it's not so bad if you already know the answer. The energy of a spring is supposed to be $E = \frac{1}{2}k\,(\Delta L)^2$ and that is what the left-hand side is supposed to become after integration.

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{2}k\left(\sqrt{x^2 + y^2} - L_F\right)^2\right] &= \frac{k}{2}\cdot 2\left(\sqrt{x^2 + y^2} - L_F\right)\frac{d}{dx}\left(\sqrt{x^2 + y^2} - L_F\right) \\ &= k\left(\sqrt{x^2 + y^2} - L_F\right)\left(\frac{1}{2}\frac{1}{\sqrt{x^2 + y^2}}\cdot 2x\right) \\ &= \frac{kx\left(\sqrt{x^2 + y^2} - L_F\right)}{\sqrt{x^2 + y^2}} \end{aligned}$$

So now we know the left-hand-side above:

$$\begin{aligned} -\frac{1}{2}k\left(\sqrt{x^2 + y^2} - L_F\right)^2 &= \frac{1}{2}mv^2 + C \\ E_0 &= \frac{1}{2}mv^2 + \frac{1}{2}k\left(\sqrt{x^2 + y^2} - L_F\right)^2 \end{aligned}$$

At the start of the problem, $x = x_0 = 8\,\text{in}$ and $v = 0$, so

$$E_0 = \frac{1}{2}k\left(\sqrt{x_0^2 + y^2} - L_F\right)^2$$

This basically gives $v\,(x)$ now (just plug in E_0 and solve for v).