Homework #8

Due in class Friday, April 25.

Unless otherwise noted, all problems on this homework are to be done without reference to calculating devices of any kind.

[**1**] In solving a particular problem, we find it convenient to introduce the variables $u = xy$ and $v = x^2 - y^2$, where x and y are the usual Cartesian coordinates.

(a) Sketch a few of the curves of constant u and v.

(b) We know that the area element in Cartesian coordinates is simple: $dA = dxdy$. Express the area element in terms of du and dv. (*Note: successfully doing this means that your answer will be of the form:* $dA = f(u,v)dudv$. *That is, there will be neither* x*s nor* y*s in your answer.*)

[**2**] Functions of a complex variable that have well-defined derivatives are called *analytic*. We know that functions are analytic only if they satisfy the *Cauchy-Riemann Equations*: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$[1].

Is the function $f(z) = \frac{1}{z}$ analytic?

[**3**] A particular object is completely covered by N patches of linear dimension L. It is found that when the linear dimension L of the patches is reduced to $\frac{1}{4}$ of its original size, the number of patches required to cover the object increases by a factor of 32.

(a) What is the dimension, D, of this object? If you obtain a ratio of logarithms for your answer, simplify it to get a numerical result.

(b) Is this object a fractal object? Explain.

[1] In case you have forgotten, the notation we are using here is the following: $f(z) = u(x,y) + iv(x,y)$ when $z = x + iy$.

[4] In a region of space free from charges, the electric potential ϕ satisfies the equation:

$$\nabla^2 \phi = 0\,.$$

(a) We will attempt to solve this equation in a square region of the xy-plane bounded by the points (a, a), $(a, -a)$, $(-a, -a)$, and $(-a, a)$. Here, a is a positive, real constant. Sketch the region of interest, along with a set of axes.

(b) Write down the two-dimensional Laplacian (∇^2) in Cartesian coordinates. Use a trial solution of the form $\phi(x, y) = f(x)g(y)$ to show that the variables in our differential equation are separable. Pass your trial solution through the differential equation, and obtain separate equations for $f(x)$ and $g(y)$.

(c) Solve your differential equations for $f(x)$ and $g(y)$, leaving any constants that you may encounter arbitrary. After you have done so, use your results to write down your general solution for $\phi(x, y)$.

(d) In this region of space, the potential is subject to the following contraints: (i) $\phi(x, y)$ vanishes eveywhere on the x-axis; (ii) the potential at point (a, a) is 10 Volts; (iii) the potential at point $(-a, -a)$ is 10 Volts. Use the constraints to fix as many constants as you can in your solution, and write down your final result for $\phi(x, y)$.

(e) You should find a solution in Part (d) that has one remaining unfixed constant. We have seen this before, and it simply indicates that we have found an infinite number of acceptable solutions. Use your favorite program to make a three-dimensional plot of $\phi(x, y)$ as a function of x and y in the given region for a few different values of your undetermined constant. For convenience, you may simply set $a = 1$ in your plots. Print and attach your plot to your homework.